

A Scale Variational Principle of Herglotz

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Abstract

The Herglotz problem is a generalization of the fundamental problem of the calculus of variations. In this paper, we consider a class of non-differentiable functions, where the dynamics is described by a scale derivative. Necessary conditions are derived to determine the optimal solution for the problem. Some other problems are considered, like transversality conditions, the multi-dimensional case, higher-order derivatives and for several independent variables.

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1 Introduction

The calculus of variations deals with optimization of a given functional, whose algebraic expression is the integral of a given function, that depends on time, space and the velocity of the trajectory:

$$x \mapsto \int_a^b L(t, x(t), \dot{x}(t)) dt.$$

The variational principle of Herglotz can be seen as an extension of such classical theories, but instead of an integral, we have the functional as a solution of a differential equation (see [9, 10]):

$$\begin{cases} \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), & \text{with } t \in [a, b], \\ z(a) = z_a. \end{cases}$$

Without the dependence of z , we can convert this problem into a calculus of variations problem. In fact, integrating the differential equation

$$\dot{z}(t) = L(t, x(t), \dot{x}(t))$$

from a to b , we obtain

$$z(b) = \int_a^b \left[L(t, x(t), \dot{x}(t)) + \frac{z_a}{b-a} \right] dt.$$

Recently, more advances were made namely proving Noether's type theorems for the variational principle of Herglotz (see e.g. [5, 6, 7, 8, 9, 12]). The aim of this paper is to consider the Herglotz problem, but the trajectories $x(\cdot)$ may be non-differentiable functions. We believe that this situation may model more efficiently certain physical problems, like fractals.

The organization of the paper is the following. In Section 2 we define what is a scale derivative, following the concept as presented in [2], and we present some of its main properties, like the algebraic rules, integration by parts formula, etc. In Section 3 we prove our new results. After presenting the Herglotz scale problem, we prove a necessary condition that every extremizer must fulfill. Some generalizations of the main result are also presented to complete the study.

2 Scale calculus

We review some definitions and the main results from [2] that we will need. For more on the subject, see references [1, 2, 3].

From now on, let α, β, h be reals in $]0, 1[$ with $\alpha + \beta > 1$ and $h \ll 1$, and consider $I := [a - h, b + h]$.

Definition 1. Let $f : I \rightarrow \mathbb{R}$ be a function. The delta derivative of f at t is defined by

$$\Delta_h[f](t) := \frac{f(t+h) - f(t)}{h}, \quad \text{for } t \in [a-h, b],$$

and the nabla derivative of f at t is defined by

$$\nabla_h[f](t) := \frac{f(t) - f(t-h)}{h}, \quad \text{for } t \in [a, b+h].$$

If f is differentiable, then

$$\lim_{h \rightarrow 0} \Delta_h[f](t) = \lim_{h \rightarrow 0} \nabla_h[f](t) = f'(t).$$

These two operators can be combined into a single one, where the real part is the mean value of such operators, and the complex part measures the difference between them.

Definition 2. The h -scale derivative of f at t is given by

$$\frac{\square_h f}{\square t}(t) = \frac{1}{2} [(\Delta_h[f](t) + \nabla_h[f](t)) + i(\Delta_h[f](t) - \nabla_h[f](t))], \quad \text{for } t \in [a, b]. \quad (1)$$

For complex valued functions f , such definition is extended by

$$\frac{\square_h f}{\square t}(t) = \frac{\square_h \operatorname{Re} f}{\square t}(t) + i \frac{\square_h \operatorname{Im} f}{\square t}(t).$$

We now explain how to drop the dependence on the parameter h in the definition of the scale derivative. First, consider the set $C_{conv}^0([a, b] \times]0, 1[, \mathbb{C})$ of the functions $g \in C^0([a, b] \times]0, 1[, \mathbb{C})$ for which the limit

$$\lim_{h \rightarrow 0} g(t, h)$$

exists for all $t \in [a, b]$, and let E be a complementary space of $C_{conv}^0([a, b] \times]0, 1[, \mathbb{C})$ in $C^0([a, b] \times]0, 1[, \mathbb{C})$.

Define π the projection of $C_{conv}^0([a, b] \times]0, 1[, \mathbb{C}) \oplus E$ onto $C_{conv}^0([a, b] \times]0, 1[, \mathbb{C})$,

$$\begin{aligned} \pi : C_{conv}^0([a, b] \times]0, 1[, \mathbb{C}) \oplus E &\rightarrow C_{conv}^0([a, b] \times]0, 1[, \mathbb{C}) \\ g := g_{conv} + g_E &\mapsto \pi(g) = g_{conv}. \end{aligned}$$

Using these definitions, we arrive at the main concept of [2].

Definition 3. The scale derivative of $f \in C^0(I, \mathbb{C})$, denoted by $\frac{\square f}{\square t}$, is defined by

$$\frac{\square f}{\square t}(t) := \left\langle \frac{\square_h f}{\square t} \right\rangle(t), \quad t \in [a, b], \quad (2)$$

where

$$\left\langle \frac{\square_h f}{\square t} \right\rangle(t) := \lim_{h \rightarrow 0} \pi \left(\frac{\square_h f}{\square t}(t) \right).$$

Definition 4. Given $f : I^n = [a - nh, b + nh] \rightarrow \mathbb{C}$, define the higher-order scale derivative of f by

$$\frac{\square^n f}{\square t^n}(t) = \frac{\square}{\square t} \left(\frac{\square^{n-1} f}{\square t^{n-1}} \right)(t), \quad t \in [a, b],$$

where $\frac{\square f^1}{\square t^1} := \frac{\square f}{\square t}$ and $\frac{\square f^0}{\square t^0} := f$.

We will adopt the notation $\square^n f(t)$ instead of $\frac{\square^n f}{\square t^n}(t)$ when there is no danger of confusion. Scale partial derivatives are also considered here. They are defined as in the standard case.

Definition 5. Let $f : \prod_{i=1}^n [a_i - h, b_i + h] \rightarrow \mathbb{R}$ be a function. Define, for each $i \in \{1, \dots, n\}$,

$$\Delta_h^i[f](t_1, \dots, t_n) := \frac{f(t_1, \dots, t_{i-1}, t_i + h, t_{i+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n)}{h},$$

for $t_i \in [a_i - h, b_i]$ and for $t_j \in [a_j - h, b_j + h]$ if $j \neq i$, and

$$\nabla_h^i[f](t_1, \dots, t_n) := \frac{f(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, t_i - h, t_{i+1}, \dots, t_n)}{h},$$

for $t_i \in [a_i, b_i + h]$ and for $t_j \in [a_j - h, b_j + h]$, if $j \neq i$. The h -scale partial derivative of f with respect to the i -th coordinate is given by

$$\frac{\square_h f}{\square t_i}(t_1, \dots, t_n) = \frac{1}{2} [(\Delta_h^i[f] + \nabla_h^i[f]) + i(\Delta_h^i[f] - \nabla_h^i[f])],$$

for $t_i \in [a_i, b_i]$.

The definition of partial scale derivatives $\square f / \square t_i$ is clear.

In what follows, we will denote

$$C_{\square}^n([a, b], \mathbb{K}) := \{f \in C^0(I^n, \mathbb{K}) \mid \frac{\square^k f}{\square t^k} \in C^0(I^{n-k}, \mathbb{C}), k = 1, 2, \dots, n\}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}.$$

Definition 6. Let $f \in C^0(I, \mathbb{C})$ and $\alpha \in]0, 1[$. We say that f is Hölderian of Hölder exponent α if there exists a constant $C > 0$ such that, for all $s, t \in I$,

$$|f(t) - f(s)| \leq C|t - s|^\alpha,$$

and we write $f \in H^\alpha(I, \mathbb{C})$, or simply $f \in H^\alpha$ when there is no danger of mislead.

We say that $f(t_1, \dots, t_n) \in H^\alpha$ if $f(t_1, \dots, t_{i-1}, \cdot, t_{i+1}, \dots, t_n) \in H^\alpha$, for all $i \in \{1, \dots, n\}$ and for all $t_j \in [a_j, b_j]$, $j \neq i$.

Theorem 1. For all $f \in H^\alpha$ and $g \in H^\beta$, we have

$$\frac{\square(f \cdot g)}{\square t}(t) = \frac{\square f}{\square t}(t) \cdot g(t) + f(t) \cdot \frac{\square g}{\square t}(t), \quad t \in [a, b].$$

Theorem 2. Let $f \in C_{\square}^1([a, b], \mathbb{R})$ be such that

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h f}{\square t} \right)_E(t) dt = 0, \quad (3)$$

where $\frac{\square_h f}{\square t} := \left(\frac{\square_h f}{\square t} \right)_{conv} + \left(\frac{\square_h f}{\square t} \right)_E$. Then,

$$\int_a^b \frac{\square f}{\square t}(t) dt = f(b) - f(a).$$

As a consequence, we have the following integration by parts formula. If

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h(f \cdot g)}{\square t} \right)_E(t) dt = 0,$$

where $f \in H^\alpha$ and $g \in H^\beta$, then

$$\int_a^b \frac{\square f}{\square t}(t) \cdot g(t) dt = [f(t)g(t)]_a^b - \int_a^b f(t) \cdot \frac{\square g}{\square t}(t) dt.$$

3 The scale variational principle of Herglotz

The (classical) variational principle of Herglotz is described in the following way. Consider the differential equation

$$\begin{cases} \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), & \text{with } t \in [a, b] \\ z(a) = z_a \\ x(a) = x_a, x(b) = x_b, \end{cases}$$

where x, z and L are smooth functions. We wish to find x (and the correspondent solution z of the system) such that $z(b)$ attains an extremum. The necessary condition is a second-order differential equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial z} \frac{\partial L}{\partial \dot{x}},$$

for all $t \in [a, b]$. This can be seen as an extension of the basic problem of calculus of variations. If L does not depend on z , then integrating the differential equation along the interval $[a, b]$, we get

$$\begin{cases} \int_a^b \left[L(t, x(t), \dot{x}(t)) + \frac{z_a}{b-a} \right] dt \rightarrow \text{extremize} \\ x(a) = x_a, x(b) = x_b. \end{cases}$$

As is well known, many physical phenomena are characterized by non-differentiable functions (e.g. generic trajectories of quantum mechanics [4], scale-relativity without the hypothesis of space-time differentiability [11]). The usual procedure is to replace the classical derivative by a scale derivative, and consider the space of continuous (and non-differentiable) functions. The scale calculus of variations approach was studied in [1, 2, 3] for a certain concept of scale derivative $\square x(t)$:

$$\begin{cases} \int_a^b L(t, x(t), \square x(t)) dt \rightarrow \text{extremize} \\ x(a) = x_a, x(b) = x_b. \end{cases}$$

Motivated by this problem, we define the fundamental scale variational principle of Herglotz. First we need to define what extremum is.

Definition 7. We say that $z \in C^1([a, b], \mathbb{C})$ attains an extremum at $t = b$ if $z'(b) = 0$.

The problem is then stated in the following way. Consider the system

$$\begin{cases} \dot{z}(t) = L(t, x(t), \square x(t), z(t)), & \text{with } t \in [a, b] \\ z(a) = z_a \\ x(a) = x_a, x(b) = x_b. \end{cases} \quad (4)$$

For simplicity, define

$$[x, z](t) := (t, x(t), \square x(t), z(t)).$$

We assume that

1. the trajectories x are in $H^\alpha \cap C_\square^1([a, b], \mathbb{R})$, $\square x \in H^\alpha$ and the functional z in $C^2([a, b], \mathbb{C})$,
2. for each x , there exists a unique solution z of the system (4)
3. z_a, x_a, x_b are fixed numbers,
4. the Lagrangian $L : [a, b] \times \mathbb{R} \times \mathbb{C}^2 \rightarrow \mathbb{C}$ is of class C^2 .

Observe that the solution $z(t)$ actually is a function on three variables, to know $z = z(t, x(t), \square x(t))$. When there is no danger of mislead, we will simply write $z(t)$. We are interested in finding a trajectory x for which the corresponding solution z is such that $z(b)$ attains an extremum. In particular, what necessary conditions such solutions must fulfill. These equations are called Euler-Lagrange

equation types. Again, problem (4) can be reduced to the scale variational problem in case L is independent of z :

$$\int_a^b L \left[(t, x(t), \square x(t)) + \frac{z_a}{b-a} \right] dt \rightarrow \text{extremize.}$$

Theorem 3. *If the pair (x, z) is a solution of problem (4), and $\frac{\partial L}{\partial \square x}[x, z] \in H^\alpha(I, \mathbb{C})$ ($\alpha \in]0, 1[$), then (x, z) is a solution of the equation*

$$\frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x}[x, z](t) \right) = \frac{\partial L}{\partial x}[x, z](t) + \frac{\partial L}{\partial z}[x, z](t) \frac{\partial L}{\partial \square x}[x, z](t), \quad (5)$$

for all $t \in [a, b]$.

Proof. Let ϵ be an arbitrary real, and consider variation functions of x of type $x(t) + \epsilon \eta(t)$, with $\eta \in H^\beta(I, \mathbb{R}) \cap C_\square^1([a, b], \mathbb{R})$ ($\beta \in]0, 1[$), $\eta(a) = \eta(b) = \square \eta(a) = 0$, and

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square x}[x, z](t) \eta(t) \right) \right) dt = 0.$$

The corresponding rate of change of z , caused by the change of x in the direction of η , is given by

$$\theta(t) = \frac{d}{d\epsilon} z(t, x(t) + \epsilon \eta(t), \square x(t) + \epsilon \square \eta(t))|_{\epsilon=0}.$$

Then

$$\begin{aligned} \dot{\theta}(t) &= \frac{d}{dt} \frac{d}{d\epsilon} z(t, x(t) + \epsilon \eta(t), \square x(t) + \epsilon \square \eta(t))|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} L(t, x(t) + \epsilon \eta(t), \square x(t) + \epsilon \square \eta(t), z(t, x(t) + \epsilon \eta(t), \square x(t) + \epsilon \square \eta(t))|_{\epsilon=0} \\ &= \frac{\partial L}{\partial x}[x, z](t) \eta(t) + \frac{\partial L}{\partial \square x}[x, z](t) \square \eta(t) + \frac{\partial L}{\partial z}[x, z](t) \theta(t). \end{aligned}$$

We obtain a first order linear differential equation on θ , whose solution is

$$\lambda(b)\theta(b) - \theta(a) = \int_a^b \lambda(t) \left[\frac{\partial L}{\partial x}[x, z](t) \eta(t) + \frac{\partial L}{\partial \square x}[x, z](t) \square \eta(t) \right] dt,$$

where

$$\lambda(t) = \exp \left(- \int_a^t \frac{\partial L}{\partial z}[x, z](\tau) d\tau \right).$$

Using the fact that $\theta(a) = \theta(b) = 0$, we get

$$\int_a^b \lambda(t) \left[\frac{\partial L}{\partial x}[x, z](t) \eta(t) + \frac{\partial L}{\partial \square x}[x, z](t) \square \eta(t) \right] dt = 0.$$

Integrating by parts the second term, we obtain

$$\int_a^b \left[\lambda(t) \frac{\partial L}{\partial x}[x, z](t) - \frac{\square}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square x}[x, z](t) \right) \right] \eta(t) dt + \left[\eta(t) \lambda(t) \frac{\partial L}{\partial \square x}[x, z](t) \right]_a^b = 0.$$

Since $\eta(a) = \eta(b) = 0$, and η is an arbitrary function elsewhere,

$$\lambda(t) \frac{\partial L}{\partial x}[x, z](t) - \frac{\square}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square x}[x, z](t) \right) = 0,$$

for all $t \in [a, b]$. Since the function $t \mapsto \lambda(t)$ is differentiable, and the function $t \mapsto \frac{\partial L}{\partial \square x}[x, z](t)$ is in H^α , it follows that

$$\lambda(t) \left(\frac{\partial L}{\partial x}[x, z](t) + \frac{\partial L}{\partial z}[x, z](t) \frac{\partial L}{\partial \square x}[x, z](t) - \frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x}[x, z](t) \right) \right) = 0.$$

Finally, since $\lambda(t) > 0$, for all t , we get

$$\frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x} [x, z](t) \right) = \frac{\partial L}{\partial x} [x, z](t) + \frac{\partial L}{\partial z} [x, z](t) \frac{\partial L}{\partial \square x} [x, z](t),$$

for all $t \in [a, b]$. \square

Remark 1. Assume that the set of state functions x is $C^1([a, b], \mathbb{R})$. Then equation (5) becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} [x, z](t) \right) = \frac{\partial L}{\partial x} [x, z](t) + \frac{\partial L}{\partial z} [x, z](t) \frac{\partial L}{\partial \dot{x}} [x, z](t),$$

which is the generalized variational principle of Herglotz as in [10].

Theorem 4. Let the pair (x, z) be a solution of the problem (4), but now $x(b)$ is free. Then (x, z) is a solution of the equation

$$\frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x} [x, z](t) \right) = \frac{\partial L}{\partial x} [x, z](t) + \frac{\partial L}{\partial z} [x, z](t) \frac{\partial L}{\partial \square x} [x, z](t),$$

for all $t \in [a, b]$, and verifies the transversality condition

$$\frac{\partial L}{\partial \square x} [x, z](b) = 0.$$

Proof. Following the proof of Theorem 3, the Euler-Lagrange equation is deduced. Then

$$\left[\eta(t) \lambda(t) \frac{\partial L}{\partial \square x} [x, z](t) \right]_a^b = 0.$$

Since $\eta(a) = 0$ and $\eta(b)$ is arbitrary, we obtain the transversality condition. \square

Multi-dimensional case

For simplicity, we considered so far one state function x only, but the multi-dimensional case (x_1, \dots, x_n) is easily studied.

Theorem 5. Let $\alpha \in]0, 1[$ and let the vector (x_1, \dots, x_n, z) be a solution of the problem: find (x_1, \dots, x_n) that extremizes $z(b)$, with

$$\begin{cases} \dot{z}(t) = L(t, x_1(t), \dots, x_n(t), \square x_1(t), \dots, \square x_n(t), z(t)), & \text{with } t \in [a, b] \\ z(a) = z_a \\ x_i(a) = x_{ia}, x_i(b) = x_{ib} \end{cases} \quad (6)$$

where, for all $i \in \{1, \dots, n\}$,

1. the trajectories x_i are in $H^\alpha \cap C_\square^1([a, b], \mathbb{R})$, $\square x_i \in H^\alpha$ and the functional z in $C^2([a, b], \mathbb{C})$,
2. z_a, x_{ia}, x_{ib} are fixed numbers,
3. $\frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z] \in H^\alpha(I, \mathbb{C})$
4. the Lagrangian $L : [a, b] \times \mathbb{R}^n \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is of class C^2 .

Then, for all $i \in \{1, \dots, n\}$, (x_1, \dots, x_n, z) is a solution of the equation

$$\frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z](t) \right) = \frac{\partial L}{\partial x_i} [x_1, \dots, x_n, z](t) + \frac{\partial L}{\partial z} [x_1, \dots, x_n, z](t) \frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z](t),$$

for all $t \in [a, b]$.

Theorem 6. Let the vector (x_1, \dots, x_n, z) be a solution of the problem as stated in Theorem 5, but now $x_i(b)$ is free, for all $i \in \{1, \dots, n\}$. Then, for all $i \in \{1, \dots, n\}$, (x_1, \dots, x_n, z) is a solution of the equation

$$\frac{\square}{\square t} \left(\frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z](t) \right) = \frac{\partial L}{\partial x_i} [x_1, \dots, x_n, z](t) + \frac{\partial L}{\partial z} [x_1, \dots, x_n, z](t) \frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z](t),$$

for all $t \in [a, b]$, and verifies the transversality condition

$$\frac{\partial L}{\partial \square x_i} [x_1, \dots, x_n, z](b) = 0.$$

Higher-order derivatives case

Theorem 7. Let $\alpha \in]0, 1[$ and let the pair (x, z) be a solution of the problem: find x that extremizes $z(b)$, with

$$\begin{cases} \dot{z}(t) = L(t, x, \square x(t), \dots, \square^n x(t), z(t)), & \text{with } t \in [a, b] \\ z(a) = z_a \\ \square^i x(a) = x_{ia}, \square^i x(b) = x_{ib}, & \text{for all } i \in \{0, \dots, n-1\}, \end{cases}$$

where

1. the trajectories x are in $H^\alpha \cap C_\square^n([a, b], \mathbb{R})$, $\square x \in H^\alpha$ and the functional z in $C^2([a, b], \mathbb{C})$,
2. z_a, x_{ia}, x_{ib} are fixed numbers, for all $i \in \{0, \dots, n-1\}$,
3. $\frac{\partial L}{\partial \square^i x} [x, z] \in H^\alpha(I^n, \mathbb{C})$, for all $i \in \{1, \dots, n\}$,
4. $[x, z](t) = (t, x, \square x(t), \dots, \square^n x(t), z(t))$ and $[x](t) = (t, x, \square x(t), \dots, \square^n x(t))$,
5. the Lagrangian $L : [a, b] \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is of class C^2 .

Then, (x, z) is a solution of the equation

$$\lambda(t) \frac{\partial L}{\partial x} [x, z](t) + \sum_{i=1}^n (-1)^i \frac{\square^i}{\square t^i} \left(\lambda(t) \frac{\partial L}{\partial \square^i x} [x, z](t) \right) = 0,$$

for all $t \in [a, b]$.

Proof. Let $x(t) + \epsilon \eta(t)$ be a variation function of x , with $\epsilon \in \mathbb{R}$ and $\eta \in H^\beta \cap C_\square^n([a, b], \mathbb{R})$ ($\beta \in]0, 1[$). Also, assume that the variations fulfill the conditions:

1. for all $i = 0, \dots, n-1$, $\square^i \eta(a) = \square^i \eta(b) = 0$, and $\square^n \eta(a) = 0$,
2. for all $i = 1, 2, \dots, n$ and $k = 0, 1, \dots, i-1$,

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h}{\square t} \left(\lambda(t) \frac{\square^k}{\square t^k} \left(\frac{\partial L}{\partial \square^i x} [x, z](t) \right) \square^{i-k-1} \eta(t) \right) \right) dt = 0.$$

Define

$$\theta(t) = \frac{d}{d\epsilon} z(t, x(t) + \epsilon \eta(t), \square x(t) + \epsilon \square \eta(t), \dots, \square^n x(t) + \epsilon \square^n \eta(t))|_{\epsilon=0}.$$

Then

$$\dot{\theta}(t) = \frac{\partial L}{\partial x} [x, z](t) \eta(t) + \sum_{i=1}^n \frac{\partial L}{\partial \square^i x} [x, z](t) \square^i \eta(t) + \frac{\partial L}{\partial z} [x, z](t) \theta(t).$$

Solving this linear ODE, we arrive at

$$\int_a^b \lambda(t) \left[\frac{\partial L}{\partial x} [x, z](t) \eta(t) + \sum_{i=1}^n \frac{\partial L}{\partial \square^i x} [x, z](t) \square^i \eta(t) \right] dt = 0,$$

where

$$\lambda(t) = \exp \left(- \int_a^t \frac{\partial L}{\partial z}[x, z](\tau) d\tau \right).$$

Integrating by parts n times, we obtain the following:

$$\begin{aligned} & \int_a^b \left[\lambda(t) \frac{\partial L}{\partial x}[x, z](t) + \sum_{i=1}^n (-1)^i \frac{\square^i}{\square t^i} \left(\lambda(t) \frac{\partial L}{\partial \square^i x}[x, z](t) \right) \right] \eta(t) dt \\ & + \left[\sum_{i=1}^n \sum_{k=0}^{i-1} (-1)^k \frac{\square^k}{\square t^k} \left(\lambda(t) \frac{\partial L}{\partial \square^i x}[x, z](t) \right) \square^{i-1-k} \eta(t) \right]_a^b = 0, \end{aligned}$$

and rearranging the terms, we get

$$\begin{aligned} & \int_a^b \left[\lambda(t) \frac{\partial L}{\partial x}[x, z](t) + \sum_{i=1}^n (-1)^i \frac{\square^i}{\square t^i} \left(\lambda(t) \frac{\partial L}{\partial \square^i x}[x, z](t) \right) \right] \eta(t) dt \\ & + \left[\sum_{i=1}^n \left[\sum_{k=i}^n (-1)^{k-i} \frac{\square^{k-i}}{\square t^{k-i}} \left(\lambda(t) \frac{\partial L}{\partial \square^k x}[x, z](t) \right) \right] \square^{i-1} \eta(t) \right]_a^b = 0. \end{aligned}$$

Since $\square^i \eta(a) = \square^i \eta(b) = 0$, for all $i \in \{0, \dots, n-1\}$ and η is arbitrary elsewhere, we get

$$\lambda(t) \frac{\partial L}{\partial x}[x, z](t) + \sum_{i=1}^n (-1)^i \frac{\square^i}{\square t^i} \left(\lambda(t) \frac{\partial L}{\partial \square^i x}[x, z](t) \right) = 0,$$

for all $t \in [a, b]$. □

Theorem 8. *Let the pair (x, z) be a solution of the problem as stated in Theorem 7, but now $\square^i x(b)$ is free, for all $i \in \{0, \dots, n-1\}$. Then, (x, z) is a solution of the equation*

$$\lambda(t) \frac{\partial L}{\partial x}[x, z](t) + \sum_{i=1}^n (-1)^i \frac{\square^i}{\square t^i} \left(\lambda(t) \frac{\partial L}{\partial \square^i x}[x, z](t) \right) = 0,$$

for all $t \in [a, b]$, and verifies the transversality condition

$$\sum_{k=i}^n (-1)^{k-i} \frac{\square^{k-i}}{\square t^{k-i}} \left(\lambda(t) \frac{\partial L}{\partial \square^k x}[x, z](t) \right) = 0 \quad \text{at } t = b,$$

for all $i \in \{1, \dots, n\}$.

Several independent variables case

We generalize Theorem 3 for several independent variables. First we fix some notations. The variable time is $t \in [a, b]$, $x = (x_1, \dots, x_n) \in \Omega := \prod_{i=1}^n [a_i, b_i]$ are the space coordinates and the state function is $u := u(t, x)$.

Theorem 9. *Let $\alpha \in]0, 1[$ and let the pair (u, z) be a solution of the problem: find u that extremizes $z(b)$, with*

$$\begin{cases} \dot{z}(t) = \int_{\Omega} L \left(t, x, u, \frac{\square u}{\square t}, \frac{\square u}{\square x_1}, \dots, \frac{\square u}{\square x_n}, z(t) \right) d^n x, & \text{with } t \in [a, b] \\ z(a) = z_a \\ u(t, x) \text{ takes fixed values,} & \forall t \in [a, b] \forall x \in \partial \Omega \\ u(t, x) \text{ takes fixed values,} & \forall t \in \{a, b\} \forall x \in \Omega, \end{cases} \quad (7)$$

where, for all $i \in \{1, \dots, n\}$,

1. the trajectories u are in $H^\alpha(I \times \Omega, \mathbb{R}) \cap C_\square^1([a, b] \times \Omega, \mathbb{R})$, $\frac{\square u}{\square t}, \frac{\square u}{\square x_i} \in H^\alpha([a, b] \times \Omega, \mathbb{C})$ and the functional z in $C^2([a, b], \mathbb{C})$,
2. z_a is a fixed number,
3. $d^n x = dx_1 \dots dx_n$,
4. $\frac{\partial L}{\partial \square t}[u, z], \frac{\partial L}{\partial \square x_i}[u, z] \in H^\alpha(I \times \Omega, \mathbb{C})$, where $\frac{\partial L}{\partial \square t}[u, z]$ denotes the partial derivative of L with respect to the variable $\frac{\square u}{\square t}$, and $\frac{\partial L}{\partial \square x_i}[u, z]$ denotes the partial derivative of L with respect to the variable $\frac{\square u}{\square x_i}$, and $[u, z](t) = (t, x, u, \frac{\square u}{\square t}, \frac{\square u}{\square x_1}, \dots, \frac{\square u}{\square x_n}, z(t))$,
5. $L : [a, b] \times \Omega \times \mathbb{R} \times \mathbb{C}^{n+2} \rightarrow \mathbb{C}$ is of class C^2 .

Then, (u, z) is a solution of the equation

$$\frac{\partial L}{\partial u}[u, z](t) + \frac{\partial L}{\partial \square t}[u, z](t) \int_\Omega \frac{\partial L}{\partial \square z}[u, z](t) d^n x - \frac{\square}{\square t} \left(\frac{\partial L}{\partial \square t}[u, z](t) \right) - \sum_{i=1}^n \frac{\square}{\square x_i} \left(\frac{\partial L}{\partial \square x_i}[u, z](t) \right) = 0,$$

for all $t \in [a, b]$ and for all $x \in \Omega$.

Proof. Let $u(t, x) + \epsilon \eta(t, x)$ be a variation function of u , with $\epsilon \in \mathbb{R}$ and $\eta \in H^\beta(I \times \Omega, \mathbb{R}) \cap C_\square^1([a, b] \times \Omega, \mathbb{R})$ ($\beta \in]0, 1[$). Also, assume that the variations fulfill the conditions:

1. $\eta(t, x) = 0, \quad \forall t \in [a, b] \forall x \in \partial\Omega$,
2. $\eta(t, x) = 0, \quad \forall t \in \{a, b\} \forall x \in \Omega$,
3. $\frac{\square \eta}{\square t}(a, x) = \frac{\square \eta}{\square x_i}(a, x) = 0, \quad \forall x \in \Omega$,
4. for all $i = 1, 2, \dots, n$,

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square t}[u, z](t) \eta(t) \right) \right)_E dt = 0.$$

and

$$\lim_{h \rightarrow 0} \int_a^b \left(\frac{\square_h}{\square x_i} \left(\lambda(t) \frac{\partial L}{\partial \square x_i}[u, z](t) \eta(t) \right) \right)_E dt = 0,$$

where

$$\lambda(t) = \exp \left(- \int_a^t \int_\Omega \frac{\partial L}{\partial z}[u, z](\tau) d^n x d\tau \right).$$

Let

$$\theta(t) = \frac{d}{d\epsilon} z \left(t, x, u + \epsilon \eta, \frac{\square u}{\square t} + \epsilon \frac{\square \eta}{\square t}, \frac{\square u}{\square x_1} + \epsilon \frac{\square \eta}{\square x_1}, \dots, \frac{\square u}{\square x_n} + \epsilon \frac{\square \eta}{\square x_n} \right) \Big|_{\epsilon=0}.$$

Proceeding with some calculations, we arrive at the ODE

$$\dot{\theta}(t) - \int_\Omega \frac{\partial L}{\partial z}[u, z](t) d^n x \theta(t) = \int_\Omega \frac{\partial L}{\partial u}[u, z](t) \eta + \frac{\partial L}{\partial \square t}[u, z](t) \frac{\square \eta}{\square t} + \sum_{i=1}^n \frac{\partial L}{\partial \square x_i}[u, z](t) \frac{\square \eta}{\square x_i} d^n x.$$

Solving the ODE, and taking into consideration that $\theta(a) = \theta(b) = 0$, we get

$$\int_a^b \int_\Omega \lambda(t) \left[\frac{\partial L}{\partial u}[u, z](t) \eta + \frac{\partial L}{\partial \square t}[u, z](t) \frac{\square \eta}{\square t} + \sum_{i=1}^n \frac{\partial L}{\partial \square x_i}[u, z](t) \frac{\square \eta}{\square x_i} \right] d^n x dt = 0.$$

Integrating by parts, and considering the boundary conditions over η , we get

$$\int_a^b \int_\Omega \left[\lambda(t) \frac{\partial L}{\partial u}[u, z](t) - \frac{\square}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square t}[u, z](t) \right) - \sum_{i=1}^n \frac{\square}{\square x_i} \left(\lambda(t) \frac{\partial L}{\partial \square x_i}[u, z](t) \right) \right] \eta d^n x dt = 0.$$

By the arbitrariness of η , it follows that for all $t \in [a, b]$ and for all $x \in \Omega$,

$$\lambda(t) \frac{\partial L}{\partial u}[u, z](t) - \frac{\square}{\square t} \left(\lambda(t) \frac{\partial L}{\partial \square t}[u, z](t) \right) - \sum_{i=1}^n \frac{\square}{\square x_i} \left(\lambda(t) \frac{\partial L}{\partial x_i}[u, z](t) \right) = 0.$$

Since $\lambda(t) > 0$, this condition implies that

$$\frac{\partial L}{\partial u}[u, z](t) + \frac{\partial L}{\partial \square t}[u, z](t) \int_{\Omega} \frac{\partial L}{\partial \square z}[u, z](t) d^n x - \frac{\square}{\square t} \left(\frac{\partial L}{\partial \square t}[u, z](t) \right) - \sum_{i=1}^n \frac{\square}{\square x_i} \left(\frac{\partial L}{\partial \square x_i}[u, z](t) \right) = 0,$$

for all $t \in [a, b]$ and for all $x \in \Omega$, and the theorem is proved. \square

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References

- [1] R. Almeida and D.F.M. Torres. Nondifferentiable variational principles in terms of a quantum operator. *Math. Methods Appl. Sci.* **34** (2011) 2231-2241.
- [2] J. Cresson and I. Greff. A non-differentiable Noethers theorem. *J. Math. Phys.* **52** (2011) No.2, 023513, 10 pp.
- [3] J. Cresson and I. Greff. Non-differentiable embedding of Lagrangian systems and partial differential equations. *J. Math. Anal. Appl.* **384** (2011) No. 2, 626-646.
- [4] R. Feynman and A. Hibbs. *Quantum mechanics and path integrals*, MacGraw-Hill, 1965.
- [5] B. Georgieva and R. Guenther. First Noether-type theorem for the generalized variational principle of Herglotz. *Topol. Methods Nonlinear Anal.* **20** (2002) No. 2, 261-273.
- [6] B. Georgieva and R. Guenther. Second Noether-type theorem for the generalized variational principle of Herglotz. *Topol. Methods Nonlinear Anal.* **26** (2005) No. 2, 307-314.
- [7] B. Georgieva, R. Guenther and T. Bodurov. Generalized variational principle of Herglotz for several independent variables. First Noether-type theorem, *J. Math. Phys.* **44** (2003) No. 9, 3911-3927.
- [8] R.B. Guenther, J.A. Gottsch and D.B. Kramer, The Herglotz algorithm for constructing canonical transformations, *SIAM Rev.* **38** (1996) No. 2, 287-293.
- [9] R.B. Guenther, C.M. Guenther and J.A. Gottsch, The Herglotz Lectures on Contact Transformations and Hamiltonian Systems, *Lecture Notes in Nonlinear Analysis*, Vol. 1, Juliusz Schauder Center for Nonlinear Studies, Nicholas Copernicus University, Toruń, 1996.
- [10] G. Herglotz, *Berührungstransformationen*, Lectures at the University of Göttingen, Göttingen, 1930.
- [11] L. Nottale. The theory of scale relativity. *Internat. J. Modern Phys. A* **7** (1992) No. 20, 4899-4936.

- [12] J.C. Orum, R.T. Hudspeth, W. Black and R.B. Guenther, Extension of the Herglotz algorithm to nonautonomous canonical transformations, SIAM Rev. **42** (2000) No. 1, 83–90.
- [13] S. Santos, N. Martins, D.F.M. Torres, Higher-order variational problems of Herglotz type, Vietnam Journal of Mathematics **42** (2014) No. 4, 409–419.